Differential Equations & Linear Algebra

FOURTH EDITION

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Pearson

DIFFERENTIAL EQUATIONS & LINEAR ALGEBRA

Fourth Edition

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CONTENTS

APPLICATION MODULES

The modules listed below follow the indicated sections in the text. Most provide computing projects that illustrate the corresponding text sections. Many of these modules are enhanced by the supplementary material found at the new Expanded Applications website, which can be accessed by visiting **goo.gl/BXB9k4**. For more information about the Expanded Applications, please review the Principal Features of this Revision section of the preface.

PREFACE

The evolution of the present text is based on experience teaching introductory differential equations and linear algebra with an emphasis on conceptual ideas and the use of applications and projects to involve students in active problem-solving experiences. Technical computing environments like *Maple*, *Mathematica*, MATLAB, and Python are widely available and are now used extensively by practicing engineers and scientists. This change in professional practice motivates a shift from the traditional concentration on manual symbolic methods to coverage also of qualitative and computer-based methods that employ numerical computation and graphical visualization to develop greater conceptual understanding. A bonus of this more comprehensive approach is accessibility to a wider range of more realistic applications of differential equations.

Both the conceptual and the computational aspects of such a course depend heavily on the perspective and techniques of linear algebra. Consequently, the study of differential equations and linear algebra in tandem reinforces the learning of both subjects. In this book we therefore have combined core topics in elementary differential equations with those concepts and methods of elementary linear algebra that are needed for a contemporary introduction to differential equations.

Principal Features of This Revision

This 4th edition is the most comprehensive and wide-ranging revision in the history of this text.

We have enhanced the exposition, as well as added graphics, in numerous sections throughout the book. We have also inserted new applications, including biological. Moreover we have exploited throughout the new interactive computer technology that is now available to students on devices ranging from desktop and laptop computers to smartphones and graphing calculators. While the text continues to use standard computer algebra systems such as *Mathematica*, *Maple*, and MATLAB, we have now added the Wolfram | Alpha website. In addition, this is the first edition of this book to feature Python, a computer platform that is freely available on the internet and which is gaining in popularity as an all-purpose scientific computing environment.

However, with a single exception of a new section inserted in Chapter 7 (noted below), the class-tested table of contents of the book remains unchanged. Therefore, instructors notes and syllabi will not require revision to continue teaching with this new edition.

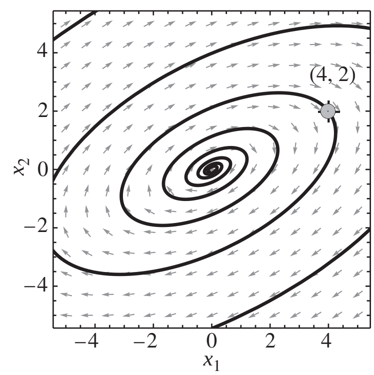
A conspicuous feature of this edition is the insertion of about 80 new computer-generated figures, many of them illustrating interactive computer applications with slider bars or touchpad controls that can be used to change initial values or parameters in a differential equation, and immediately see in real time the resulting changes in the structure of its solutions.

Some illustrations of the revisions and updating in this edition:

New Exposition In a number of sections, we have added new text and graphics to enhance student understanding of the subject matter. For instance, see the new introductory treatments of separable equations in Section 1.4 (page 30), of linear equations in Section 1.5 (page 46), and of isolated critical points in Sections 9.1 (page 503) and 9.2 (page 514). Also we have updated the examples and accompanying graphics in Sections 2.4–2.6, 7.3, and 7.7 to illustrate modern calculator technology.

New Interactive Technology and Graphics New figures throughout the text illustrate the capability that modern computing technology platforms offer to vary initial conditions and other parameters interactively. These figures are accompanied by detailed instructions that allow students to recreate the figures and make full use of the interactive features. For example, Section 7.4 includes the figure shown, a *Mathematica*-drawn phase plane diagram for a linear system of the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$; after putting the accompanying code into *Mathematica*, the user can immediately see the effect of changing the initial condition by clicking and dragging the "locator point" initially set at $(4, 2)$.

Similarly, the application module for Section 5.1 now offers MATLAB and TI-Nspire graphics with interactive slider bars that vary the coefficients of a linear differential equation. The Section 11.2 application module features a new MATLAB graphic in which the user can vary the order of a series solution of an initial value problem, again immediately displaying the resulting graphical change in the corresponding approximate solution.



New *Mathematica* graphic in Section 7.4

New Content The single entirely new section for this edition is Section 7.4, which is devoted to the construction of a "gallery" of phase plane portraits illustrating all the possible geometric behaviors of solutions of the 2-dimensional linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. In motivation and preparation for the detailed study of eigenvalue-eigenvector methods in subsequent sections of Chapter 7 (which then follow in the same order as in the previous edition), Section 7.4 shows how the particular arrangements of eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$ correspond to identifiable patterns—"fingerprints," so to speak—in the phase plane portrait of the system. The resulting gallery is shown in the two pages of phase plane portraits in Figure 7.4.16 (pages 417–418) at the end of the section. The new 7.4 application module (on dynamic phase plane portraits, page 421) shows how students can use interactive computer systems to bring to life this gallery by allowing initial conditions, eigenvalues, and even eigenvectors to vary in real time. This dynamic approach is then illustrated with several new graphics inserted in the remainder of Chapter 7.

Finally, for a new biological application, see the application module for Section 9.4, which now includes a substantial investigation (page 551) of the nonlinear FitzHugh–Nagumo equations of neuroscience, which were introduced to model the behavior of neurons in the nervous system.

New Topical Headings Many of the examples and problems are now organized under headings that make the topic easy to see at a glance. This not only adds to the readability of the book, but it also makes it easier to choose in-class examples and homework problems. For instance, most of the text examples in Section 1.4 are now labelled by topic, and the same is true of the wealth of problems following this section.

New Expanded Applications Website The effectiveness of the application modules located throughout the text is greatly enhanced by the supplementary material found at the new Expanded Applications website. Nearly all of the application modules in the text are marked with and a unique "tiny URL"—a web address that leads directly to an Expanded Applications page containing a wealth of electronic resources supporting that module. Typical Expanded Applications materials include an enhanced and expanded PDF version of the text with further discussion or additional applications, together with computer files in a variety of platforms, including *Mathematica*, *Maple*, MATLAB, and in some cases Python and/or TI calculator. These files provide all code appearing in the text as well as equivalent versions in other platforms, allowing students to immediately use the material in the Application Module on the computing platform of their choice. In addition to the URLs dispersed throughout the text, the Expanded Applications can be accessed by going to the Expanded Applications home page through this URL: `goo.gl/BXB9k4`. Note that when you enter URLs for the Extended Applications, take care to distinguish the following characters:

`l` = lowercase L	`1` = one
`I` = uppercase I	`0` = zero
`O` = uppercase O	

Features of This Text

Computing Features The following features highlight the flavor of computing technology that distinguishes much of our exposition.

- Almost 600 *computer-generated figures* show students vivid pictures of direction fields, solution curves, and phase plane portraits that bring symbolic solutions of differential equations to life.
- About three dozen *application modules* follow key sections throughout the text. Most of these applications outline technology investigations that can be carried out using a variety of popular technical computing systems and which seek to actively engage students in the application of new technology. These modules are accompanied by the new Expanded Applications website previously detailed, which provides explicit code for nearly all of the applications in a number of popular technology platforms.
- The early introduction of numerical solution techniques in Chapter 2 (on mathematical models and numerical methods) allows for a fresh numerical emphasis throughout the text. Here and in Chapter 7, where numerical techniques for systems are treated, a concrete and tangible flavor is achieved by the inclusion of numerical algorithms presented in parallel fashion for systems ranging from graphing calculators to MATLAB and Python.

Modeling Features Mathematical modeling is a goal and constant motivation for the study of differential equations. For a small sample of the range of applications in this text, consider the following questions:

- What explains the commonly observed time lag between indoor and outdoor daily temperature oscillations? (Section 1.5)
- What makes the difference between doomsday and extinction in alligator populations? (Section 2.1)

- How do a unicycle and a car react differently to road bumps? (Sections 5.6 and 7.5)
- Why might an earthquake demolish one building and leave standing the one next door? (Section 7.5)
- How can you predict the time of next perihelion passage of a newly observed comet? (Section 7.7)
- What determines whether two species will live harmoniously together or whether competition will result in the extinction of one of them and the survival of the other? (Section 9.3)

Organization and Content This text reshapes the usual sequence of topics to accommodate new technology and new perspectives. For instance:

- After a precis of first-order equations in Chapter 1 (though with the coverage of certain traditional symbolic methods streamlined a bit), Chapter 2 offers an early introduction to mathematical modeling, stability and qualitative properties of differential equations, and numerical methods—a combination of topics that frequently are dispersed later in an introductory course.
- Chapters 3 (Linear Systems and Matrices), 4 (Vector Spaces), and 6 (Eigenvalues and Eigenvectors) provide concrete and self-contained coverage of the elementary linear algebra concepts and techniques that are needed for the solution of linear differential equations and systems. Chapter 4 includes sections 4.5 (row and column spaces) and 4.6 (orthogonal vectors in $\mathbf{R}^n$). Chapter 6 concludes with applications of diagonalizable matrices and a proof of the Cayley–Hamilton theorem for such matrices.
- Chapter 5 exploits the linear algebra of Chapters 3 and 4 to present efficiently the theory and solution of single linear differential equations. Chapter 7 is based on the eigenvalue approach to linear systems, and includes (in Section 7.6) the Jordan normal form for matrices and its application to the general Cayley–Hamilton theorem. This chapter includes an unusual number of applications (ranging from railway cars to earthquakes) of the various cases of the eigenvalue method, and concludes in Section 7.7 with numerical methods for systems.
- Chapter 8 is devoted to matrix exponentials with applications to linear systems of differential equations. The spectral decomposition method of Section 8.3 offers students an especially concrete approach to the computation of matrix exponentials.
- Chapter 9 exploits linear methods for the investigation of nonlinear systems and phenomena, and ranges from phase plane analysis to applications involving ecological and mechanical systems.
- Chapters 10 treats Laplace transform methods for the solution of constant-coefficient linear differential equations with a goal of handling the piecewise continuous and periodic forcing functions that are common in physical applications. Chapter 11 treats power series methods with a goal of discussing Bessel's equation with sufficient detail for the most common elementary applications.

This edition of the text also contains over 1800 end-of-section exercises, including a wealth of application problems. The Answers to Selected Problems section (page 677) includes answers to most odd-numbered problems plus a good many even-numbered ones, as well as about 175 computer-generated graphics to enhance its value as a learning aid.

Supplements

Instructor's Solutions Manual (0-13-449825-9) is available for instructors to download at Pearson's Instructor Resource Center (pearsonhighered.com/irc). This manual provides worked-out solutions for most of the problems in the book.

Student's Solutions Manual (0-13-449814-3) contains solutions for most of the odd-numbered problems.

Both manuals have been reworked extensively for this edition with improved explanations and more details inserted in the solutions of many problems.

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1 First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Example 1 The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y. ■

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find—either exactly or approximately—the appropriate solution of that equation.
3. To interpret the solution that is found.

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a differential equation, we are challenged to find the unknown *functions* $y = y(x)$ for which an identity such as $y'(x) = 2xy(x)$—that is, the differential equation

$$\frac{dy}{dx} = 2xy$$

—holds on some interval of real numbers. Ordinarily, we will want to find *all* solutions of the differential equation, if possible.

Example 2 If C is a constant and

$$y(x) = Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = (2x)\left(Ce^{x^2}\right) = 2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx} = 2xy \tag{2}$$

for all x. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the arbitrary constant C. By the method of separation of variables (Section 1.4) it can be shown that every solution of the differential equation in (2) is of the form in Eq. (1). ■

Differential Equations and Mathematical Models

The following three examples illustrate the process of translating scientific laws and principles into differential equations. In each of these examples the independent variable is time t, but we will see numerous examples in which some quantity other than time is the independent variable.

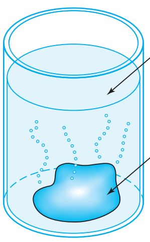

FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

Example 3 **Rate of cooling** Newton's law of cooling may be stated in this way: The *time rate of change* (the rate of change with respect to time t) of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium (Fig. 1.1.1). That is,

$$\frac{dT}{dt} = -k(T - A), \tag{3}$$

where k is a positive constant. Observe that if $T > A$, then $dT/dt < 0$, so the temperature is a decreasing function of t and the body is cooling. But if $T < A$, then $dT/dt > 0$, so that T is increasing.

Thus the physical law is translated into a differential equation. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then—with the aid of this formula—we can predict the future temperature of the body. ■

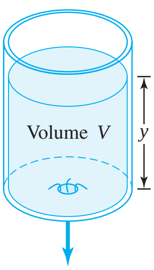

FIGURE 1.1.2. Torricelli's law of draining, Eq. (4), describes the draining of a water tank.

Example 4 **Draining tank** Torricelli's law implies that the *time rate of change* of the volume V of water in a draining tank (Fig. 1.1.2) is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dt} = -k\sqrt{y}, \tag{4}$$

where k is a constant. If the tank is a cylinder with vertical sides and cross-sectional area A, then $V = Ay$, so $dV/dt = A \cdot (dy/dt)$. In this case Eq. (4) takes the form

$$\frac{dy}{dt} = -h\sqrt{y}, \tag{5}$$

where $h = k/A$ is a constant. ■

Example 5 **Population growth** The *time rate of change* of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \tag{6}$$

where k is the constant of proportionality. ■

Let us discuss Example 5 further. Note first that each function of the form

$$P(t) = Ce^{kt} \tag{7}$$

is a solution of the differential equation

$$\frac{dP}{dt} = kP$$

in (6). We verify this assertion as follows:

$$P'(t) = Cke^{kt} = k\left(Ce^{kt}\right) = kP(t)$$

for all real numbers t. Because substitution of each function of the form given in (7) into Eq. (6) produces an identity, all such functions are solutions of Eq. (6).

Thus, even if the value of the constant k is known, the differential equation $dP/dt = kP$ has *infinitely many* different solutions of the form $P(t) = Ce^{kt}$, one for each choice of the "arbitrary" constant C. This is typical of differential equations. It is also fortunate, because it may allow us to use additional information to select from among all these solutions a particular one that fits the situation under study.

Example 6 **Population growth** Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the population at time $t = 0$ (hours, h) was 1000, and that the population doubled after 1 h. This additional information about $P(t)$ yields the following equations:

$$1000 = P(0) = Ce^0 = C,$$
$$2000 = P(1) = Ce^k.$$

It follows that $C = 1000$ and that $e^k = 2$, so $k = \ln 2 \approx 0.693147$. With this value of k the differential equation in (6) is

$$\frac{dP}{dt} = (\ln 2)P \approx (0.693147)P.$$

Substitution of $k = \ln 2$ and $C = 1000$ in Eq. (7) yields the particular solution

$$P(t) = 1000e^{(\ln 2)t} = 1000(e^{\ln 2})^t = 1000 \cdot 2^t \qquad (\text{because } e^{\ln 2} = 2)$$

that satisfies the given conditions. We can use this particular solution to predict future populations of the bacteria colony. For instance, the predicted number of bacteria in the population after one and a half hours (when $t = 1.5$) is

$$P(1.5) = 1000 \cdot 2^{3/2} \approx 2828.$$ ■

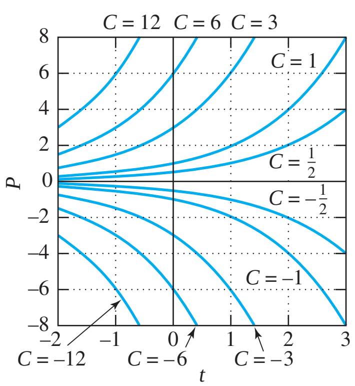

FIGURE 1.1.3. Graphs of $P(t) = Ce^{kt}$ with $k = \ln 2$.

The condition $P(0) = 1000$ in Example 6 is called an **initial condition** because we frequently write differential equations for which $t = 0$ is the "starting time." Figure 1.1.3 shows several different graphs of the form $P(t) = Ce^{kt}$ with $k = \ln 2$. The graphs of all the infinitely many solutions of $dP/dt = kP$ in fact fill the entire two-dimensional plane, and no two intersect. Moreover, the selection of any one point P_0 on the P-axis amounts to a determination of $P(0)$. Because exactly one solution passes through each such point, we see in this case that an initial condition $P(0) = P_0$ determines a unique solution agreeing with the given data.

Mathematical Models

Our brief discussion of population growth in Examples 5 and 6 illustrates the crucial process of *mathematical modeling* (Fig. 1.1.4), which involves the following:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation—for example, answering the question originally posed.

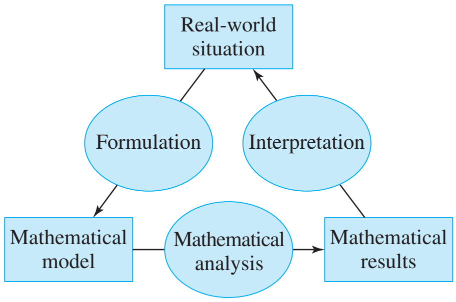

FIGURE 1.1.4. The process of mathematical modeling.

In the population example, the real-world problem is that of determining the population at some future time. A **mathematical model** consists of a list of variables (P and t) that describe the given situation, together with one or more equations relating these variables ($dP/dt = kP$, $P(0) = P_0$) that are known or are assumed to hold. The mathematical analysis consists of solving these equations (here, for P as a function of t). Finally, we apply these mathematical results to attempt to answer the original real-world question.

As an example of this process, think of first formulating the mathematical model consisting of the equations $dP/dt = kP$, $P(0) = 1000$, describing the bacteria population of Example 6. Then our mathematical analysis there consisted of solving for the solution function $P(t) = 1000e^{(\ln 2)t} = 1000 \cdot 2^t$ as our mathematical result. For an interpretation in terms of our real-world situation—the actual bacteria population—we substituted $t = 1.5$ to obtain the predicted population of $P(1.5) \approx 2828$ bacteria after 1.5 hours. If, for instance, the bacteria population is growing under ideal conditions of unlimited space and food supply, our prediction may be quite accurate, in which case we conclude that the mathematical model is adequate for studying this particular population.

On the other hand, it may turn out that no solution of the selected differential equation accurately fits the actual population we're studying. For instance, for *no* choice of the constants C and k does the solution $P(t) = Ce^{kt}$ in Eq. (7) accurately describe the actual growth of the human population of the world over the past few centuries. We must conclude that the differential equation $dP/dt = kP$ is inadequate for modeling the world population—which in recent decades has "leveled off" as compared with the steeply climbing graphs in the upper half ($P > 0$) of Fig. 1.1.3. With sufficient insight, we might formulate a new mathematical model including a perhaps more complicated differential equation, one that takes into account such factors as a limited food supply and the effect of increased population on birth and death rates. With the formulation of this new mathematical model, we may attempt to traverse once again the diagram of Fig. 1.1.4 in a counterclockwise manner. If we can solve the new differential equation, we get new solution functions to compare with the real-world population. Indeed, a successful population analysis may require refining the mathematical model still further as it is repeatedly measured against real-world experience.

But in Example 6 we simply ignored any complicating factors that might affect our bacteria population. This made the mathematical analysis quite simple, perhaps unrealistically so. A satisfactory mathematical model is subject to two contradictory requirements: It must be sufficiently detailed to represent the real-world situation with relative accuracy, yet it must be sufficiently simple to make the mathematical analysis practical. If the model is so detailed that it fully represents the physical situation, then the mathematical analysis may be too difficult to carry out. If the model is too simple, the results may be so inaccurate as to be useless. Thus there is an inevitable tradeoff between what is physically realistic and what is mathematically possible. The construction of a model that adequately bridges this gap between realism and feasibility is therefore the most crucial and delicate step in the process. Ways must be found to simplify the model mathematically without sacrificing essential features of the real-world situation.

Mathematical models are discussed throughout this book. The remainder of this introductory section is devoted to simple examples and to standard terminology used in discussing differential equations and their solutions.

Examples and Terminology

Example 7 If C is a constant and $y(x) = 1/(C - x)$, then

$$\frac{dy}{dx} = \frac{1}{(C-x)^2} = y^2$$

if $x \neq C$. Thus

$$y(x) = \frac{1}{C - x} \tag{8}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2 \tag{9}$$

on any interval of real numbers not containing the point $x = C$. Actually, Eq. (8) defines a *one-parameter family* of solutions of $dy/dx = y^2$, one for each value of the arbitrary constant or "parameter" C. With $C = 1$ we get the particular solution

$$y(x) = \frac{1}{1 - x}$$

that satisfies the initial condition $y(0) = 1$. As indicated in Fig. 1.1.5, this solution is continuous on the interval $(-\infty, 1)$ but has a vertical asymptote at $x = 1$. ■

Example 8 Verify that the function $y(x) = 2x^{1/2} - x^{1/2}\ln x$ satisfies the differential equation

$$4x^2 y'' + y = 0 \tag{10}$$

for all $x > 0$.

Solution First we compute the derivatives

$$y'(x) = -\tfrac{1}{2}x^{-1/2}\ln x \quad \text{and} \quad y''(x) = \tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}.$$

Then substitution into Eq. (10) yields

$$4x^2 y'' + y = 4x^2\left(\tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2}\ln x = 0$$

if x is positive, so the differential equation is satisfied for all $x > 0$. ■

The fact that we can write a differential equation is not enough to guarantee that it has a solution. For example, it is clear that the differential equation

$$(y')^2 + y^2 = -1 \tag{11}$$

has *no* (real-valued) solution, because the sum of nonnegative numbers cannot be negative. For a variation on this theme, note that the equation

$$(y')^2 + y^2 = 0 \tag{12}$$

obviously has only the (real-valued) solution $y(x) \equiv 0$. In our previous examples any differential equation having at least one solution indeed had infinitely many.

The **order** of a differential equation is the order of the highest derivative that appears in it. The differential equation of Example 8 is of second order, those in Examples 2 through 7 are first-order equations, and

$$y^{(4)} + x^2 y^{(3)} + x^5 y = \sin x$$

is a fourth-order equation. The most general form of an ***n*th-order** differential equation with independent variable x and unknown function or dependent variable $y = y(x)$ is

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0, \tag{13}$$

where F is a specific real-valued function of $n + 2$ variables.

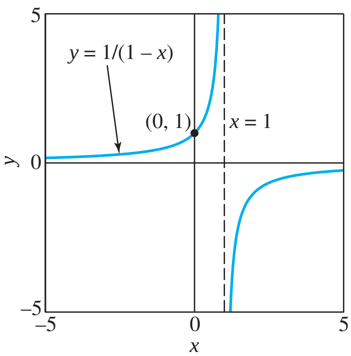

FIGURE 1.1.5. The solution of $y' = y^2$ defined by $y(x) = 1/(1 - x)$.

Our use of the word *solution* has been until now somewhat informal. To be precise, we say that the continuous function $u = u(x)$ is a **solution** of the differential equation in (13) **on the interval** I provided that the derivatives $u', u'', \dots, u^{(n)}$ exist on I and

$$F\left(x, u, u', u'', \dots, u^{(n)}\right) = 0$$

for all x in I. For the sake of brevity, we may say that $u = u(x)$ **satisfies** the differential equation in (13) on I.

Remark Recall from elementary calculus that a differentiable function on an open interval is necessarily continuous there. This is why only a continuous function can qualify as a (differentiable) solution of a differential equation on an interval. ■

Example 7 **Continued** Figure 1.1.5 shows the two "connected" branches of the graph $y = 1/(1 - x)$. The left-hand branch is the graph of a (continuous) solution of the differential equation $y' = y^2$ that is defined on the interval $(-\infty, 1)$. The right-hand branch is the graph of a *different* solution of the differential equation that is defined (and continuous) on the different interval $(1, \infty)$. So the single formula $y(x) = 1/(1 - x)$ actually defines two different solutions (with different domains of definition) of the same differential equation $y' = y^2$. ■

Example 9 If A and B are constants and

$$y(x) = A\cos 3x + B\sin 3x, \tag{14}$$

then two successive differentiations yield

$$y'(x) = -3A\sin 3x + 3B\cos 3x,$$
$$y''(x) = -9A\cos 3x - 9B\sin 3x = -9y(x)$$

for all x. Consequently, Eq. (14) defines what it is natural to call a *two-parameter family* of solutions of the second-order differential equation

$$y'' + 9y = 0 \tag{15}$$

on the whole real number line. Figure 1.1.6 shows the graphs of several such solutions. ■

Although the differential equations in (11) and (12) are exceptions to the general rule, we will see that an nth-order differential equation ordinarily has an n-parameter family of solutions—one involving n different arbitrary constants or parameters.

In both Eqs. (11) and (12), the appearance of y' as an implicitly defined function causes complications. For this reason, we will ordinarily assume that any differential equation under study can be solved explicitly for the highest derivative that appears; that is, that the equation can be written in the so-called *normal form*

$$y^{(n)} = G\left(x, y, y', y'', \dots, y^{(n-1)}\right), \tag{16}$$

where G is a real-valued function of $n + 1$ variables. In addition, we will always seek only real-valued solutions unless we warn the reader otherwise.

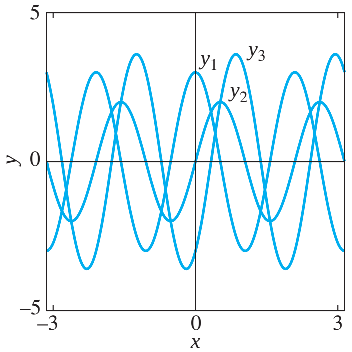

FIGURE 1.1.6. The three solutions $y_1(x) = 3\cos 3x$, $y_2(x) = 2\sin 3x$, and $y_3(x) = -3\cos 3x + 2\sin 3x$ of the differential equation $y'' + 9y = 0$.

All the differential equations we have mentioned so far are **ordinary** differential equations, meaning that the unknown function (dependent variable) depends on only a *single* independent variable. If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are, the equation is called a **partial** differential equation. For example, the temperature $u = u(x, t)$ of a long thin uniform rod at the point x at time t satisfies (under appropriate simple conditions) the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where k is a constant (called the *thermal diffusivity* of the rod). In Chapters 1 through 8 we will be concerned only with *ordinary* differential equations and will refer to them simply as differential equations.

In this chapter we concentrate on *first-order* differential equations of the form

$$\frac{dy}{dx} = f(x, y). \tag{17}$$

We also will sample the wide range of applications of such equations. A typical mathematical model of an applied situation will be an **initial value problem**, consisting of a differential equation of the form in (17) together with an **initial condition** $y(x_0) = y_0$. Note that we call $y(x_0) = y_0$ an initial condition whether or not $x_0 = 0$. To **solve** the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{18}$$

means to find a differentiable function $y = y(x)$ that satisfies both conditions in Eq. (18) on some interval containing x_0.

Example 10 Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

Solution We need only find a value of C so that the solution $y(x) = 1/(C - x)$ satisfies the initial condition $y(1) = 2$. Substitution of the values $x = 1$ and $y = 2$ in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

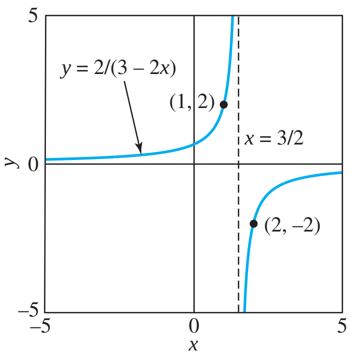

FIGURE 1.1.7. The solutions of $y' = y^2$ defined by $y(x) = 2/(3-2x)$.

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

Figure 1.1.7 shows the two branches of the graph $y = 2/(3-2x)$. The left-hand branch is the graph on $(-\infty, \frac{3}{2})$ of the solution of the given initial value problem $y' = y^2$, $y(1) = 2$. The right-hand branch passes through the point $(2, -2)$ and is therefore the graph on $(\frac{3}{2}, \infty)$ of the solution of the different initial value problem $y' = y^2$, $y(2) = -2$. ■

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually *find* or *compute* that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$
2. $y' + 2y = 0$; $y = 3e^{-2x}$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$
4. $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$
5. $y' = y + 2e^{-x}$; $y = e^x - e^{-x}$
6. $y'' + 4y' + 4y = 0$; $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$
7. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$
8. $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$
9. $y' + 2xy^2 = 0$; $y = \dfrac{1}{1 + x^2}$
10. $x^2 y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$
11. $x^2 y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$
12. $x^2 y'' - xy' + 2y = 0$; $y_1 = x\cos(\ln x)$, $y_2 = x\sin(\ln x)$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$
14. $4y'' = y$
15. $y'' + y' - 2y = 0$
16. $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$
18. $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$
19. $y' = y + 1$; $y(x) = Ce^x - 1$, $y(0) = 5$
20. $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$
21. $y' + 3x^2 y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$
22. $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$
23. $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$
24. $xy' - 3y = x^3$; $y(x) = x^3(C + \ln x)$, $y(1) = 17$
25. $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$
26. $y' + y\tan x = \cos x$; $y(x) = (x + C)\cos x$, $y(\pi) = 0$

In Problems 27 through 31, a function $y = g(x)$ is described by some geometric property of its graph. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions).

27. The slope of the graph of g at the point (x, y) is the sum of x and y.
28. The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(x/2, 0)$.
29. Every straight line normal to the graph of g passes through the point $(0, 1)$. Can you *guess* what the graph of such a function g might look like?
30. The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.
31. The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

Differential Equations as Models

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population P is proportional to the square root of P.
33. The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.
34. The acceleration dv/dt of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

35. In a city having a fixed population of P persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.

36. In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37. $y'' = 0$ **38.** $y' = y$

39. $xy' + y = 3x^2$ **40.** $(y')^2 + y^2 = 1$

41. $y' + y = e^x$ **42.** $y'' + y = 0$

Problems 43 through 46 concern the differential equation

$$\frac{dx}{dt} = kx^2,$$

where k is a constant.

43. **(a)** If k is a constant, show that a general (one-parameter) solution of the differential equation is given by $x(t) = 1/(C - kt)$, where C is an arbitrary constant.

(b) Determine by inspection a solution of the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** Assume that k is positive, and then sketch graphs of solutions of $x' = kx^2$ with several typical positive values of $x(0)$.

(b) How would these solutions differ if the constant k were negative?

45. Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. Based on the result of Problem 43, how long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

46. Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $dv/dt = kv^2$. The initial speed of the motorboat is $v(0) = 10$ meters per second (m/s), and v is decreasing at the rate of 1 m/s^2 when $v = 5$ m/s. Based on the result of Problem 43, long does it take for the velocity of the boat to decrease to 1 m/s? To $\frac{1}{10}$ m/s? When does the boat come to a stop?

47. In Example 7 we saw that $y(x) = 1/(C - x)$ defines a one-parameter family of solutions of the differential equation $dy/dx = y^2$. **(a)** Determine a value of C so that $y(10) = 10$. **(b)** Is there a value of C such that $y(0) = 0$? Can you nevertheless find by inspection a solution of $dy/dx = y^2$ such that $y(0) = 0$? **(c)** Figure 1.1.8 shows typical graphs of solutions of the form $y(x) = 1/(C - x)$. Does it appear that these solution curves fill the entire xy-plane? Can you conclude that, given any point (a, b) in the plane, the differential equation $dy/dx = y^2$ has exactly one solution $y(x)$ satisfying the condition $y(a) = b$?

48. **(a)** Show that $y(x) = Cx^4$ defines a one-parameter family of differentiable solutions of the differential equation $xy' = 4y$ (Fig. 1.1.9). **(b)** Show that

$$y(x) = \begin{cases} -x^4 & \text{if } x < 0, \\ x^4 & \text{if } x \geqq 0 \end{cases}$$

defines a differentiable solution of $xy' = 4y$ for all x, but is not of the form $y(x) = Cx^4$. **(c)** Given any two real numbers a and b, explain why—in contrast to the situation in part (c) of Problem 47—there exist infinitely many differentiable solutions of $xy' = 4y$ that all satisfy the condition $y(a) = b$.

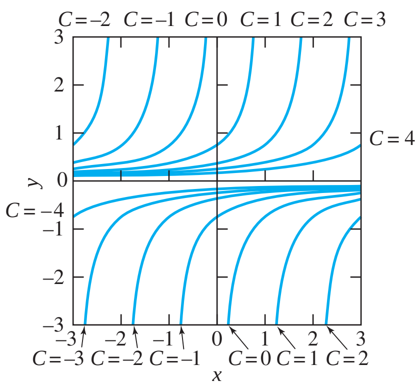

FIGURE 1.1.8. Graphs of solutions of the equation $dy/dx = y^2$.

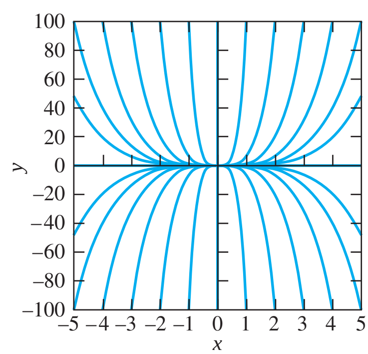

FIGURE 1.1.9. The graph $y = Cx^4$ for various values of C.